Containment Problem for points on another reducible conic

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Recall:

**Q:** Given an ideal $I$, how do $I^{(m)}$ and $I^r$ compare?

If $I \subseteq R = k[\mathbb{P}^N]$ is the ideal of points $p_1, p_2, \ldots, p_d \in \mathbb{P}^N$, then $I^{(m)} = \cap_i I(p_i)^m$.

**Example:** If $I$ defines a complete intersection, $I^{(m)} = I^m$.

**Facts:** Given $0 \neq I \subsetneq R = k[\mathbb{P}^N]$ homogeneous,

- $I^r \subseteq I^{(m)} \iff r \geq m$.
- $I^{(m)} \subseteq I^r \Rightarrow m \geq r$, so assume $m \geq r$.

**Containment Problem (CP):** For which $m$ and $r$ is $I^{(m)} \subseteq I^r$?
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**Facts:** Given \( 0 \neq I \subsetneq R = k[\mathbb{P}^N] \) homogeneous,
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- \( I^{(m)} \subseteq I^r \implies m \geq r \), so assume \( m \geq r \).

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Two ways to solve the CP

- (Exact Solution) Find the set of all \((m, r)\) such that \(I^{(m)} \subseteq I^r\).

- (Asymptotic Solution) Find the resurgence, \(\rho(I)\), defined by Bocci and Harbourne:

\[
\rho(I) = \sup \left\{ \frac{m}{r} : I^{(m)} \nsubseteq I^r \right\}
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Obviously, \(m/r > \rho(I)\) implies \(I^{(m)} \subseteq I^r\).
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Some facts about $\rho(I)$

**Theorem (Hochster-Huneke)**

For $I \subseteq k[\mathbb{P}^N]$ homogeneous, $I^{(rN)} \subseteq I^r$.

**Corollary**

If $I \subseteq k[\mathbb{P}^N]$ is nontrivial and homogeneous, $1 \leq \rho(I) \leq N$.

For ideals $0 \neq I \subsetneq k[\mathbb{P}^N]$ homogeneous,
- If $I$ defines a complete intersection, then $\rho(I) = 1$.
- No $I$ is known with $\rho(I) = N$.
- Computing $\rho$ is hard; complete solutions are even harder.
- Exact values of $\rho$ are known in only a few cases.

Today: $I$ defines points on a pair of lines.
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Points in $\mathbb{P}^2$, ideals in $R = k[\mathbb{P}^2] = k[x, y, z]$:

\[
\begin{align*}
  y &= 0, \\
  x &= 0
\end{align*}
\]

- $n \geq 3$
- $I(p_0) = (x, y)$ and $I(p_1 + \cdots + p_n) = (z, F)$, $F \in k[x, y]$, $\deg F = n$
- $I = I(p_0 + p_1 + \cdots + p_n) = (x, y) \cap (z, F) = (xz, yz, F)$
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The idea
Find compatible $k$-bases of the ideals

**Theorem ($k$-basis of $R$)**

$R = k[x, y, z]$ is spanned by “monomials” of the form $x^eF^iy^jz^l$, where $0 \leq e < n$.

Idea of the proof: $R = \text{span}_k \langle x^\beta y^\gamma z^\delta \rangle$ and deg $F = n$, so replace $x^{bn}$ with $F^b$.

**Proposition ($k$-bases of $(z, F)^m$ and $(x, y)^m$)**

(a) The ideal $(z, F)^m$ is spanned by forms $x^eF^iy^jz^l$ satisfying $e + in + ln \geq mn$ for $e, i, j, l \geq 0$.

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Corollary \((k\)-basis of \(I^{(m)}\))

Let \(m \geq 1\). Recall \(I^{(m)} = (x, y)^m \cap (z, F)^m\). Then \(I^{(m)}\) is spanned by “monomials” of the form \(x^e F^i y^j z^l\), where \(e, i, j, l \geq 0, 0 \leq e < n\) and

(a) \(e + in + ln \geq mn\), and

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Proposition \((k\)-basis of \(I^r\))

Let \(r \geq 1\). Then \(I^r\) is spanned by elements of the form \(x^e F^i y^j z^l\) with \(e, i, j, l \geq 0\) and:

(a) \(l < j\) and \(e + in + nl \geq rn\), or

(b) \(e + in + j \leq l\) and \(e + in + j \geq r\), or

(c) \(j \leq l < e + in + j\) and \(e + in + j + (n - 1)l \geq rn\).
Corollary (\(k\)-basis of \(I^{(m)}\))

Let \(m \geq 1\). Recall \(I^{(m)} = (x, y)^m \cap (z, F)^m\). Then \(I^{(m)}\) is spanned by “monomials” of the form \(x^e F^i y^j z^l\), where \(e, i, j, l \geq 0\), \(0 \leq e < n\) and

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Corollary ($k$-basis of $I^{(m)}$)

Let $m \geq 1$. Recall $I^{(m)} = (x, y)^m \cap (z, F)^m$. Then $I^{(m)}$ is spanned by “monomials” of the form $x^e F^i y^j z^l$, where $e, i, j, l \geq 0$, $0 \leq e < n$ and

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Let $r \geq 1$. Then $I^r$ is spanned by elements of the form $x^e F^i y^j z^l$ with $e, i, j, l \geq 0$ and:

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(c) $j \leq l < e + in + j$ and $e + in + j + (n - 1)l \geq rn$. 
An Example

Claim: \( I^{(7)} \not\subseteq I^6 \) when \( n = \deg F = 3 \).

Consider \( xF^2z^5 \in I^{(7)} \) since it satisfies the inequalities

(a) \( e + in + ln \geq mn \) (i.e., \( 1 + 2 \cdot 3 + 5 \cdot 3 \geq 7 \cdot 3 \)), and

(b) \( e + in + j \geq m \) (i.e., \( 1 + 2 \cdot 3 + 0 \geq 7 \))

but \( xF^2z^5 \not\in I^6 \) since

- \( j \leq l < e + in + j \) (i.e., \( 0 \leq 5 < 1 + 2 \cdot 3 + 0 \))
- but \( e + in + j + (n - 1)l \geq rn \) (i.e., \( 1 + 2 \cdot 3 + 0 + (3 - 1) \cdot 5 = 17 \geq 18 = 6 \cdot 3 \)) fails.
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but $xF^2z^5 \notin I^6$ since

- $j \leq l < e + in + j$ (i.e., $0 \leq 5 < 1 + 2 \cdot 3 + 0$)
- but $e + in + j + (n - 1)l \geq rn$
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  (i.e., \( 1 + 2 \cdot 3 + 0 + (3 - 1) \cdot 5 = 17 \geq 18 = 6 \cdot 3 \)) fails.
Main Results

Two solutions to the CP

Theorem (Complete Solution)

Let $I$ be the ideal of $n \geq 3$ collinear points and one point off the line. Then $I^{(m)} \nsubseteq I'$ holds if and only if either

- $m < n$ and $m \leq \frac{rn^2 + rn - n - 2}{n^2}$, or
- $m \geq n$ and $m \leq \frac{n^2 r - n}{n^2 - n + 1}$.

Theorem (Asymptotic Solution)

For the ideal $I$ of $n \geq 3$ collinear points and one point off the line,

$$\rho(I) = \frac{n^2}{n^2 - n + 1}.$$
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**Theorem (Asymptotic Solution)**

For the ideal $I$ of $n \geq 3$ collinear points and one point off the line,

$$\rho(I) = \frac{n^2}{n^2 - n + 1}.$$
Computing $\rho(I)$ from the complete solution

Assume $m \geq n$

Recall $I^{(m)} \not\subseteq I^r \iff m \leq \frac{n^2 r - n}{n^2 - n + 1}$.

Then $\frac{m}{r} > \frac{n^2}{n^2 - n + 1} > \frac{n^2 - n/r}{n^2 - n + 1} \Rightarrow m > \frac{n^2 r - n}{n^2 - n + 1} \Rightarrow I^{(m)} \subseteq I^r$.

Thus, $\rho(I) \leq \frac{n^2}{n^2 - n + 1}$.

Conversely, take $m = tn^2 - 1$ and $r = t(n^2 - n + 1)$. Then

$m \leq \frac{n^2 r - n}{n^2 - n + 1}$, but

$$\lim_{t \to \infty} \frac{m}{r} = \lim_{t \to \infty} \frac{tn^2 - 1}{t(n^2 - n + 1)} = \frac{n^2}{n^2 - n + 1},$$

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Symbolic powers are ordinary powers

Theorem

If $I$ is the ideal of $n$ collinear points and one point off the line, then $I^{(nt)} = (I^{(n)})^t$ for all $t \geq 1$. Moreover, $n$ is the least positive integer for which this equality holds for all $t \geq 1$.

As a consequence, the symbolic power algebra $\oplus I^{(m)}$ is Noetherian.
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As a consequence, the symbolic power algebra $\bigoplus I^{(m)}$ is Noetherian.
Two conjectures

Harbourne and Huneke conjectured:

**Conjecture**

\[ I^{(2r)} \subseteq M^r I^r, \text{ where } M \text{ is the ideal generated by the variables.} \]

**Conjecture**

\[ I^{(2r-1)} \subseteq M^{r-1} I^r, \text{ where } M \text{ is the ideal generated by the variables.} \]

Both are true for the ideal \( I \) of \( n \) collinear points and one point off the line.
Thank you!