1 Sequences

DEF: An ordered field is a field $F$ and total order $<\ (\forall x, y, z \in F)$:
(i) $x < y, y < x \Rightarrow x = y$,
(ii) $x < y, y < z \Rightarrow x < z$
(iii) $x < y \Rightarrow x + z < y + z$
(iv) $0 < y, x \Rightarrow 0 < xy$.  

DEF: The Archimedean property on an ordered field $F$ is $\forall x, y \in F\ x, y > 0\ \exists N \in \mathbb{N}$ such that $n \cdot x > y$.

FACT: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} + \frac{c}{d} = \frac{a}{b}$ for all $a, b, c, d, x, s \in \mathbb{Z}$ with $rb + sd \neq 0$.

DEF: A real number $L$ is the limit of a sequence of real numbers $(a_n)_{n=1}^\infty$ if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Then, $(a_n)$ converges to $L$.

THM: “Squeeze Theorem” Suppose three sequences $(a_n), (b_n), (c_n)$ satisfy $a_n \leq b_n \leq c_n$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$. Then $\lim_{n \to \infty} b_n = L$.

PROP: If $(a_n)_{n=1}^\infty$ converges, then the set $\{a_n | n \in \mathbb{N}\}$ is bounded.

THM: If $\lim_{n \to \infty} a_n = L$, $\lim_{n \to \infty} b_n = M$, and $a \in \mathbb{R}$, then
(i) $\lim_{n \to \infty} a_n + b_n = L + M$,
(ii) $\lim_{n \to \infty} a_n a_n = aL$,
(iii) $\lim_{n \to \infty} a_n b_n = LM$,
(iv) $\lim_{n \to \infty} a_n b_n = \frac{L}{M}$ if $M \neq 0$.

THM: “Least Upper Bound Principle” Every nonempty subset $S$ of $\mathbb{R}$ that is bounded above has a supremum. Similarly, every nonempty subset $S$ of $\mathbb{R}$ that is bounded below has an infimum.

THM: “Monotone Convergence Theorem” A monotone increasing sequence that is bounded above converges. A monotone decreasing sequence that is bounded below converges.

THM: Let $(a_n)$ be a sequence.
(i) If $\lim a_n$ is defined, then $\lim \inf a_n = \lim \sup a_n = \lim a_n$.
(ii) If $\lim \inf a_n = \lim \sup a_n = L$, then $\lim a_n = L$.

DEF: A subsequence $(a_{n_k})_{k=1}^\infty$ is a new sequence $(a_{n_k})_{k=1}^\infty = (a_{n_1}, a_{n_2}, \ldots)$ where $n_1 < n_2 < \cdots$.

THM: If the sequence $(a_n)$ converges, then every subsequence converges to the same limit.

THM: Every sequence $(a_n)$ has a monotonic subsequence.

COR: Let $(a_n)$ be a sequence. There exists a monotonic subsequence whose limit is $\limsup a_n$ and there exists a monotonic subsequence shows limit is $\liminf a_n$.

DEF: Let $(a_n)$ be a sequence in $\mathbb{R}$. A subsequential limit is any real number (or symbol $+\infty, -\infty$) that is the limit of some subsequence $(a_{n_k})$.

THM: Let $(a_n)$ be any sequence in $\mathbb{R}$, and let $S$ denote the set of subsequential limits of $(a_n)$.
(i) $S$ is nonempty.
(ii) $\sup S = \lim \sup a_n$ and $\inf S = \lim \inf a_n$.
(iii) $\lim a_n$ exists if and only if $S$ has a single element, namely $\lim a_n$.

THM: Let $S$ denote the set of subsequential limits of a sequence $(a_n)$. Suppose $(b_n)$ is a sequence in $S \cap \mathbb{R}$ and that $t = \lim b_n$. Then $t \in S$.

LMA: “Nested Intervals Lemma” Suppose that $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$ are nonempty closed intervals such that $I_{n+1} \subseteq I_n$ for each $n \geq 1$. Then the intersection $\bigcap_{n \geq 1} I_n$ is nonempty.

THM: “Bolzano-Weierstrass Theorem” Every bounded sequence of real numbers has a convergent subsequence.

FACT: For a bounded sequence $(a_n)$, $\lim \sup a_n (\lim \inf a_n)$ is the largest (smallest) possible value for a convergent subsequence.

DEF: A sequence $(a_n)_{n=1}^\infty$ is called a Cauchy sequence if for every $\varepsilon > 0$, there is an integer $N$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq N$.

DEF: A subset $S$ of $\mathbb{R}$ is said to be complete if every Cauchy sequence converges to a point in $S$.

THM: “Completeness Theorem” A sequence of real numbers converges if and only if it is a Cauchy sequence. In particular, $\mathbb{R}$ is complete.

2 Series

DEF: Given a sequence $(a_n)_{n=1}^\infty$, the infinite series $\sum_{n=1}^\infty a_n = \lim_{n \to \infty} \sum_{k=1}^n a_k$ converges if the limit exists, diverges otherwise.

FACT: Some series can be solved using a telescoping sum, by cancelling elements between sequence terms.

THM: “nth term test” If $\lim a_n \neq 0$, then $\sum a_n$ diverges.
**Thm:** “Cauchy Criterion for Series” The following are equivalent for a series ∑ₙ₌₁ⁿ₌₀ aₙ.

(i) The series converges.
(ii) For every ε > 0, there is an N ∈ ℕ so that |∑ₙ₌₁ⁿ₌₀+₁ aₙ| < ε for all n ≥ N.
(iii) For every ε > 0, there is an N ∈ ℕ so that |∑ₙ₌₁ⁿ₌₀ m₌₁ᵐ₌₀+₁ aₙ| < ε if n, m ≥ N.  

**Prop:** If aₙ ≥ 0 for k ≥ 1 and sₙ = ∑ₙ₌₁ⁿ₌₀ aₙ, then either

(i) (sₙ)ₙ₌₁ⁿ₌₀ is bounded above, in which case ∑ₙ₌₁ⁿ₌₀ aₙ converges.
(ii) (sₙ)ₙ₌₁ⁿ₌₀ is unbounded, in which case ∑ₙ₌₁ⁿ₌₀ aₙ diverges.

**Def:** A sequence (aₙ)ₙ₌₀ is a **geometric series** with ratio r if aₙ₊₁ = rₙaₙ for all n ≥ 0, or equivalently aₙ = a₀rⁿ.

**Thm:** “Convergence of Geometric Series” A geometric series ∑ₙ₌₁ⁿ₌₀ aₙ converges to 1/(1−r) if |r| < 1.

**Def:** A sequence (aₙ)ₙ₌₁ⁿ₌₀ is a **p-series** with power p if aₙ = 1/ⁿᵖ for some a, p ∈ R.

**Thm:** “Convergence of p-series” A p-series ∑ₙ₌₁ⁿ₌₀ 1/ⁿᵖ converges if and only if p > 1.

**Thm:** “Comparison Test” Consider two sequences (aₙ), (bₙ) with |aₙ| ≤ bₙ for all n ≥ 1. If (bₙ) is summable, then (aₙ) is summable and |∑ₙ₌₁ⁿ₌₀ aₙ| ≤ ∑ₙ₌₁ⁿ₌₀ bₙ. If (aₙ) is not summable, then (bₙ) is not summable.

**Thm:** “Ratio Test” A series ∑ₙ₌₀ aₙ of nonzero terms

(i) Converges absolutely if lim sup |aₙ₊₁/aₙ| < 1,
(ii) Diverges if lim inf |aₙ₊₁/aₙ| > 1.

**Thm:** “Root Test” Suppose that aₙ ≥ 0 for all n and let ℓ = lim sup √ⁿ√ /aₙ. If ℓ < 1, then ∑ₙ₌₁ⁿ₌₀ aₙ converges absolutely, and if ℓ > 1, the series diverges.

**Thm:** “Integral Test” If f : R → R is positive and decreasing, then ∫ₐ₌₁ⁿ₌₀ f(n) converges if and only if limᵣ₌₁₋ₐ f(n)dx exists (and is finite).

**Thm:** “Limit Comparison Test” If ∑ₙ₌₁ⁿ₌₀ bₙ is a convergent series of positive numbers bₙ and lim sup |aₙ|/bₙ < ∞ then ∑ₙ₌₁ⁿ₌₀ aₙ converges.

**Def:** A sequence is **alternating** if it has the form (−1)ⁿaₙ or (−1)ⁿ⁺¹aₙ where aₙ ≥ 0 for all n ≥ 1.

**Thm:** “Leibniz Alternating Series Test” Suppose that (aₙ)ₙ₌₁ⁿ₌₀ is a monotone decreasing sequence a₁ ≥ a₂ ≥ a₃ ≥ ⋯ ≥ 0 and that limₙ₌₁ⁿ₌₀ aₙ = 0. Then the alternating series ∑ₙ₌₁ⁿ₌₀ (−1)ⁿaₙ converges.

**Cor:** Suppose that (aₙ)ₙ₌₁ⁿ₌₀ is a monotone decreasing positive sequence and that limₙ₌₁ⁿ₌₀ aₙ = 0. Then the difference between the sum of the alternating series ∑ₙ₌₁ⁿ₌₀ (−1)ⁿaₙ and the Nth partial sum is at most |aₙ|.

**Def:** A series ∑ₙ₌₁ⁿ₌₀ aₙ is called **absolutely convergent** if the series ∑ₙ₌₁ⁿ₌₀ |aₙ| converges. A series that converges but is not absolutely convergent is called **conditionally convergent**.

**Def:** A **rearrangement** of a series ∑ₙ₌₁ⁿ₌₀ aₙ is another series with the same terms in a different order. This can be described by a permutation π of the natural numbers N determining the series ∑ₙ₌₁ⁿ₌₀ aₙ(π(n)).

**Thm:** For an absolutely convergent series, every rearrangement converges to the same limit.

**Lma:** Let ∑ₙ₌₁ⁿ₌₀ aₙ be a convergent series. Denote the positive terms as b₁, b₂, … and the other terms as c₁, c₂, ….

(i) If ∑ₙ₌₁ⁿ₌₀ aₙ is absolutely convergent, then so are both ∑ₙ₌₁ⁿ₌₀ bₙ and ∑ₙ₌₁ⁿ₌₀ |cₙ|, and ∑ₙ₌₁ⁿ₌₀ aₙ = ∑ₙ₌₁ⁿ₌₀ bₙ − ∑ₙ₌₁ⁿ₌₀ |cₙ|.
(ii) If ∑ₙ₌₁ⁿ₌₀ aₙ is conditionally convergent, then ∑ₙ₌₁ⁿ₌₀ bₙ and ∑ₙ₌₁ⁿ₌₀ |cₙ| both diverge.

**Thm:** “Rearrangement Theorem” If ∑ₙ₌₁ⁿ₌₀ aₙ is a conditionally convergent series, then for every real number L, there is a rearrangement that converges to L.

**Lma:** “Summation by Parts Lemma” Suppose (xₙ) and (yₙ) are sequences of real numbers. Define Xₙ = ∑ₙ₌₁ⁿ₌₀ xₙ and Yₙ = ∑ₙ₌₁ⁿ₌₀ yₙ. Then ∑ₙ₌₁ⁿ₌₀ xₙ₊₁ yₙ₊₁ = XₙYₙ₊₁ − Xₙ₊₁ Yₙ₊₁.

**Thm:** “Dirichlet’s Test” Suppose that (aₙ)ₙ₌₁ⁿ₌₀ is a sequence of real numbers with bounded partial sums. If (bₙ)ₙ₌₁ⁿ₌₀ is a sequence of positive numbers decreasing monotonically to 0, then the series ∑ₙ₌₁ⁿ₌₀ aₙbₙ converges.

**Thm:** “Abel’s Test” Suppose that ∑ₙ₌₁ⁿ₌₀ aₙ converges and (bₙ) is a monotonic convergent sequence. Then, ∑ₙ₌₁ⁿ₌₀ aₙbₙ converges.

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3 **Topology of Rⁿ**

**Def:** The **dot product** or **inner product** of two vectors x, y ∈ Rⁿ is ⟪x, y⟫ = ∑ₙ₌₁ⁿ₌₀ xᵣyᵣ.  

**Thm:** “Schwarz Inequality” For all x, y ∈ Rⁿ, | ⟪x, y⟫ | ≤ |x| ⋅ |y|. Equality holds if and only if x and y are colinear.

**Thm:** “Triangle Inequality” The triangle inequality holds for the Euclidean norm on Rⁿ: |x + y| ≤ |x| + |y| for all x, y ∈ Rⁿ. Moreover, equality holds if and only if either x = 0 or y = cx with c ≥ 0.
DEF: A set $V = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^n$ is orthonormal if $\langle v_i, v_j \rangle = 0$ when $i \neq j$ and $\langle v_i, v_i \rangle = 1$. If $m = 0$, then $V$ spans $\mathbb{R}^n$ and is called an orthonormal basis.$^{50}$

LMA: Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in $\mathbb{R}^n$. Then $\sum_{i=1}^n |v_i|^2 = 1$.$^{51}$

DEF: A sequence of points $(x_k)$ in $\mathbb{R}^n$ converges to a point $a$ if for every $\varepsilon > 0$, there is an integer $N$ so that $\|x_k - a\| < \varepsilon$ for all $k > N$. In this case, write $\lim_{k \to \infty} x_k = a$.$^{52}$

LMA: Let $(x_k)$ be a sequence in $\mathbb{R}^n$. Then $\lim_{k \to \infty} x_k = a$ if and only if $\lim_{k \to \infty} |x_k - a| = 0$.$^{53}$

LMA: A sequence $x_k = (x_{k,1}, \ldots, x_{k,n})$ in $\mathbb{R}^n$ converges to a point $a = (a_1, \ldots, a_n)$ if and only if each coordinate converges: $\lim_{k \to \infty} x_{k,i} = a_i$ for $1 \leq i \leq n$.$^{54}$

DEF: A sequence $x_k$ in $\mathbb{R}^n$ is Cauchy if for every $\varepsilon > 0$, there is an integer $N$ so that $\|x_k - x_\ell\| < \varepsilon$ for all $k, \ell > N$. A set $S \subseteq \mathbb{R}^n$ is complete if every Cauchy sequence of points in $S$ converges to a point in $S$.$^{55}$

THM: “Completeness Theorem for $\mathbb{R}^n$” Every Cauchy sequence in $\mathbb{R}^n$ converges. Thus, $\mathbb{R}^n$ is complete.$^{56}$

DEF: A point $x$ is a limit point of a subset $A \subseteq \mathbb{R}^n$ if there is a sequence $(a_k)_{k=1}^\infty$ with $a_k \in A$ such that $x = \lim_{k \to \infty} a_k$.

DEF: A set $A \subseteq \mathbb{R}^n$ is closed if it contains all of its limit points.$^{57}$

DEF: A point $x$ is a cluster point of a subset $A \subseteq \mathbb{R}^n$ if there is a sequence $(a_n)_{n=1}^\infty$ with $a_n \in A \setminus \{x\}$ such that $x = \lim_{n \to \infty} a_n$.$^{58}$

PROP: If $A, B \subseteq \mathbb{R}^n$ are closed, then $A \cup B$ is closed. If $\{A_i \mid i \in I\}$ is a family of closed subsets of $\mathbb{R}^n$, then $\cap_{i \in I} A_i$ is closed.$^{59}$

E.G.: Let $A_n = \overline{B_{\frac{1}{2^n}}}(0)$. The family of sets $\{A_n\}_{n \in \mathbb{N}}$ has every set closed, but $\cap_{n \in \mathbb{N}} A_n = B_1(0)$, which is not closed.$^{60}$

DEF: If $A$ is a subset of $\mathbb{R}^n$, the closure of $A$ is the set $\overline{A}$ consisting of all limit points of $A$.$^{61}$

DEF: The ball about $a$ in $\mathbb{R}^n$ of radius $r$ is the set $B_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\}$. A subset $U \subseteq \mathbb{R}^n$ is open if for every $a \in U$, there is some $r > 0$ so that the ball $B_r(a)$ is contained in $U$.$^{62}$

PROP: Let $A \subseteq \mathbb{R}^n$. Then $\overline{A}$ is the smallest closed set containing $A$. In particular, $\overline{\overline{A}} = \overline{A}$.$^{63}$

THM: “Duality of Open and Closed Sets” A set $A \subseteq \mathbb{R}^n$ is open if and only if the complement of $A$, $A' = \{x \in \mathbb{R}^n \mid x \notin A\}$ is closed.$^{64}$

HERE MARKS THE END OF EXAM 1 MATERIAL—

PROP: If $U$ and $V$ are open subsets of $\mathbb{R}^n$ then $U \cap V$ is an open subset of $\mathbb{R}^n$. If $\{U_i \mid i \in I\}$ is a family of open subsets of $\mathbb{R}^n$, then $\bigcup_{i \in I} U_i$ is open.$^{65}$

E.G.: Let $A_n = \overline{B_{\frac{1}{2^n}}}(0)$. The family of sets $\{A_n\}_{n \in \mathbb{N}}$ has every set open, but $\cap_{n \in \mathbb{N}} A_n = B_1(0)$, which is closed.$^{66}$

DEF: The interior, $\text{int} X$, of a set $X$ is the largest open set contained in $X$. If $\text{int} X = \emptyset$, then $X$ has empty interior.$^{67}$

DEF: A subset $A \subseteq \mathbb{R}^n$ is compact if every sequence $(a_k)_{k=1}^\infty$ of points in $A$ has a convergent subsequence $(a_{k_i})_{i=1}^\infty$ with limit $a = \lim_{i \to \infty} a_{k_i}$ in $A$.$^{68}$

DEF: A subset $S$ of $\mathbb{R}^n$ is called bounded provided that there is a real number $R$ such that $S$ is contained in the ball $B_R(0)$.$^{69}$

LMA: A compact subset of $\mathbb{R}^n$ is closed and bounded.$^{70}$

LMA: If $C$ is a closed subset of a compact subset of $\mathbb{R}^n$, then $C$ is compact.$^{71}$

LMA: The cube $[a, b]^n$ is a compact subset of $\mathbb{R}^n$.$^{72}$

THM: “Heine-Borel Theorem” A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.$^{73}$

THM: “Cantor’s Intersection Theorem” If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a decreasing sequence of nonempty compact subsets of $\mathbb{R}^n$, then $\cap_{k \geq 1} A_k$ is not empty.$^{74}$

DEF: A set whose closure has no interior is nowhere dense. A point $x$ of a set $A$ is isolated if there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(x)$ intersects $A$ only in the singleton $\{x\}$. A set $A$ is perfect if each point $x \in A$ is the limit of some sequence in $A \setminus \{x\}$.$^{75}$

4 Functions

DEF: Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$. If $a \in S$ is a cluster point (limit point of $S \setminus \{a\}$) then a point $v \in \mathbb{R}^n$ is a limit of $f$ at $a$ if for every $\varepsilon > 0$ there is an $r > 0$ so that $\|f(x) - v\| < \varepsilon$ whenever $0 < \|x - a\| < r$ and $x \in S$. Write $\lim_{x \to a} f(x) = v$.$^{76}$

DEF: Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$. $f$ is continuous at $a \in S$ if for every $\varepsilon > 0$, there is an $r > 0$ such that, for all $x \in S$ with $\|x - a\| < r$, we have $\|f(x) - f(a)\| < \varepsilon$. Moreover, $f$ is continuous on $S$ if it is continuous at each point $a \in S$. If $f$ is not continuous at $a$, $f$ is discontinuous at $a$.$^{77}$

Mike Janssen
Def: A function \( f : S \to \mathbb{R}^m \) is called \textit{Lipschitz} if there is a constant \( C \) such that \( \| f(x) - f(a) \| \leq C \| x - a \| \) for all \( x, a \in S \). The \textit{Lipschitz constant} of \( f \) is the smallest choice of \( C \) for this condition to hold.\(^{78}\)

Def: A function \( f : [a, b] \to \mathbb{R} \) satisfies a \textit{Lipschitz condition of order} \( \alpha > 0 \) if there is some positive constant \( M \) so that \( |f(x_1) - f(x_2)| \leq M |x_1 - x_2|^{\alpha} \).\(^{79}\)

Prop: Every Lipschitz function is continuous.\(^{80}\)

Cor: Every linear transformation \( A \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is Lipschitz and therefore continuous.\(^{81}\)

Def: The \textit{coordinate functions} are, \( \pi_j : \mathbb{R}^n \to \mathbb{R} \) is given as \( \pi_j(x_1, \ldots, x_n) = x_j \) for \( 1 \leq j \leq n \), and \( \epsilon_i(t) = te_i \).\(^{82}\)

Def: A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) has a \textit{removable singularity} at \( a \) if \( \lim_{x \to a} f(x) \) exists, but does not equal \( f(a) \).\(^{83}\)

Def: The \textit{limit of} \( f \) as \( x \) approaches \( a \) \textit{from the right} is \( L \) if for every \( \varepsilon > 0 \), there is an \( \delta > 0 \) so that \( |f(x) - L| < \varepsilon \) for all \( a < x < a + \delta \), written \( \lim_{x \to a^+} f(x) = L \). Similarly, define the limit from the left, \( \lim_{x \to a^-} f(x) = L \).\(^{84}\)

Def: When a function \( f \) on \( R \) has limits from the left and right that are different, we say \( f \) has a \textit{jump discontinuity}.

A function on an interval is called \textit{piecewise continuous} if on every finite subinterval, it has only a finite number of jump discontinuities and is continuous at all other points.\(^{85}\)

E.G.: The \textit{Heaviside function} \( H(x) \) is defined to be 0 for all \( x < 0 \) and 1 for \( x \geq 0 \).\(^{86}\)

E.G.: The \textit{ceiling function} \( \lceil x \rceil \) has countably many jump discontinuities.\(^{87}\)

E.G.: For any subset \( A \subseteq \mathbb{R}^n \), the \textit{characteristic function} of \( A \) is \( \chi_A(x) \), which equals 1 if \( x \in A \), and 0 otherwise.\(^{88}\)

E.G.: Let \( f(x) = 0 \) when \( x \not\in \mathbb{Q} \) and then \( f(x) = \frac{1}{x^2} \) when \( x = \frac{p}{q} \), in reduced form. Then, \( f(x) \) is continuous only at irrational points, and discontinuous on the rationals.\(^{89}\)

Def: Say that the limit of a function \( f(x) \) has \( x \) approaches \( a \) \textit{as} \( +\infty \) if for every positive integer \( N \), there is an \( r > 0 \) so that \( f(x) > N \) for all \( 0 < |x - a| < r \). We write \( \lim_{x \to a} f(x) = +\infty \). We define the limit \( \lim_{x \to a} f(x) = -\infty \) similarly.\(^{90}\)

Def: A function \( f \) is \textit{asymptotic to the curve} \( g \) if \( \lim_{x \to \infty} |f(x) - g(x)| = 0 \).\(^{91}\)

Def: A subset \( V \subseteq S \subseteq \mathbb{R}^n \) is \textit{open} in \( S \) (or \textit{relatively open}) if there is an open set \( U \) in \( \mathbb{R}^n \) such that \( U \cap S = V \).\(^{92}\)

Thm: For a function \( f \) mapping \( S \subseteq \mathbb{R}^n \) into \( \mathbb{R}^m \), the following are equivalent:

(i) \( f \) is continuous on \( S \).

(ii) “Sequential characterization of continuity” For every convergent sequence \( (x_n)_{n=1}^{\infty} \) with \( \lim_{n \to \infty} x_n = a \in S \), \( \lim_{n \to \infty} f(x_n) = f(a) \).

(iii) “Topological characterization of continuity” For every open set \( U \) in \( \mathbb{R}^m \), the set \( f^{-1}(U) = \{ x \in S \mid f(x) \in U \} \) is open in \( S \).\(^{93}\)

Thm: If \( f, g \) are functions from a common domain \( S \) into \( \mathbb{R}^m \), \( a \in S \) such that \( \lim_{x \to a} f(a) = u \) and \( \lim_{x \to a} g(x) = v \), then

(i) \( \lim_{x \to a} f(x) + g(x) = u + v \).

(ii) \( \lim_{x \to a} af(x) = au \).

When the range is contained in \( \mathbb{R} \), say \( \lim_{x \to a} f(x) = u \) and \( \lim_{x \to a} g(x) = v \), then

(iii) \( \lim_{x \to a} f(x)g(x) = uv \), and

(iv) \( \lim_{x \to a} f(x)/g(x) = u/v \) provided \( v \neq 0 \).\(^{94}\)

Thm: If \( f, g \) are functions form \( S \) to \( \mathbb{R}^m \) that are continuous at \( a \in S \) and \( \alpha \in \mathbb{R} \), then

(i) \( f + \alpha g \) is continuous at \( a \).

(ii) \( \alpha f \) is continuous at \( a \).

When the range is contained in \( \mathbb{R} \),

(iii) \( fg \) is continuous at \( a \).

(iv) \( f/g \) is continuous at \( a \) provided that \( g(a) \neq 0 \).\(^{95}\)

Def: A function \( f \) is a \textit{rational function} if \( f(x) = p(x)/q(x) \) where \( p, q \in \mathbb{R}[x] \) and \( q \neq 0 \). Rational functions are continuous except where \( q(x) = 0 \).\(^{96}\)

Def: If a function \( f : S \to T \) and \( g : T \to \mathbb{R}^m \), then the \textit{composition} of \( g \) and \( f \), denoted \( g \circ f \) is the function that sends \( x \) to \( g(f(x)) \).\(^{97}\)

Thm: Suppose \( f : S \to T \) and \( g : T \to \mathbb{R}^m \). If \( f \) is continuous at \( a \in S \) and \( g \) is continuous at \( f(a) \in T \), then \( g \circ f \) is continuous at \( a \).\(^{98}\)

Thm: Let \( C \) be a compact subset of \( \mathbb{R}^n \), and let \( f \) be a continuous function from \( C \) into \( \mathbb{R}^m \). Then the image set \( f(C) \) is compact.\(^{99}\)

Thm: “Extreme Value Theorem” Let \( C \) be a compact subset of \( \mathbb{R}^n \) and let \( f \) be a continuous function from \( C \) into \( \mathbb{R} \). Then there are points \( a \) and \( b \) in \( C \) attaining the minimum and maximum values of \( f \) on \( C \). That is, \( f(a) \leq f(x) \leq f(b) \) for all \( x \in C \).\(^{100}\)

Mike Janssen
5 Intro to Calculus

DEF: A function \( f : \mathbb{R} \to \mathbb{R}^m \) is a periodic function if there exists a \( d > 0 \) such that \( f(x) = f(x + d) \) for all \( x \in \mathbb{R} \). The least such \( d \) is called the period of \( f \). Then, \( f \) is \( d \)-periodic.

DEF: A function \( f : S \to \mathbb{R}^m \) is uniformly continuous if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that \( ||f(x) - f(a)|| < \varepsilon \) whenever \( |x - a| < \delta \) for \( x, a \in S \).

PROP: Every Lipschitz function is uniformly continuous.

COR: Every linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is uniformly continuous.

THM: Suppose that \( C \subseteq \mathbb{R}^n \) is compact and \( f : C \to \mathbb{R}^n \) is continuous. Then \( f \) is uniformly continuous on \( C \).

THM: “Intermediate Value Theorem” If \( f \) is a continuous real-valued function on \([a, b]\) with \( f(a) < 0 < f(b) \), then there exists a point \( c \in (a, b) \) such that \( f(c) = 0 \).

COR: If \( f \) is a continuous real-valued function on \([a, b]\), then \( f([a, b]) \) is a closed interval.

DEF: A path in \( S \subseteq \mathbb{R}^n \) from \( a \) to \( b \), both points in \( S \), is the image of a continuous function \( \gamma \) from \([0, 1]\) into \( S \) such that \( \gamma(0) = a \) and \( \gamma(1) = b \).

COR: Suppose that \( S \subseteq \mathbb{R}^n \) and \( f \) is a continuous real-valued function on \( S \). If there is a path from \( a \) to \( b \) in \( S \) and \( f(a) < 0 < f(b) \), then there is a point \( c \) on the path so that \( f(c) = 0 \).

DEF: A function \( f \) is increasing on an interval \((a, b)\) if \( f(x) \leq f(y) \) whenever \( a < x \leq y < b \). It is strictly increasing on \((a, b)\) if \( f(x) < f(y) \) whenever \( a < x < y < b \). Similarly, define decreasing and strictly decreasing functions. All of these functions are called monotone.

PROP: If \( f \) is an increasing function on the interval \([a, b]\), then the one-sided limits of \( f \) exist at each point \( c \in (a, b) \) and \( \lim_{x \to c^-} f(x) = L < f(x) \leq \lim_{x \to c^+} f(x) = M \). For decreasing functions, the inequalities are reversed.

COR: The only type of discontinuity that a monotone function on an interval can have is a jump discontinuity.

COR: If \( f \) is a monotone function on \([a, b]\) and the range of \( f \) intersects every nonempty open interval in \([f(a), f(b)]\) then \( f \) is continuous.

THM: A monotone function on \([a, b]\) has at most countably many discontinuities.

THM: Let \( f \) be a continuous strictly increasing function on \([a, b]\). Then \( f \) maps \([a, b]\) one-to-one and onto \([f(a), f(b)]\).

Moreover the inverse function \( f^{-1} \) is also continuous and strictly increasing.

PROP: Let \( f : S \to \mathbb{R}^m, S \subseteq \mathbb{R}^n \) be a continuous function. If \( T \subset S \) is compact, \( f(T) \) is compact.

E.G.: The Cantor function \( c : [0, 1] \to [0, 1] \) is an onto function defined such that \( c \) is constant over the intervals not in the Cantor set, but is strictly increasing over the Cantor set. Also, \( c(C) = [0, 1] \).

5 Intro to Calculus

DEF: A function \( f : (a, b) \to \mathbb{R} \) is differentiable at a point \( x_0 \in (a, b) \) if \( \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \) exists. We write \( f'(x_0) \) for this limit.

DEF: When \( f \) is differentiable at \( x_0 \), we define the tangent line to \( f \) at \( x_0 \) to be the linear function \( T(x) = f(x_0) + f'(x_0)(x - x_0) \).

PROP: If \( f \) is differentiable at \( x_0 \), then it is continuous at \( x_0 \). Differentiable functions are continuous.

LMA: Let \( f \) be a function on \([a, b]\) that is differentiable at \( x_0 \). Let \( T(x) \) be the tangent line to \( f \) at \( x_0 \). Then \( T \) is the unique linear function with the property \( \lim_{x \to x_0} \frac{f(x) - T(x)}{x - x_0} = 0 \).

COR: If \( f(x) \) is a function on \((a, b)\) and \( x_0 \in (a, b) \), then the following are equivalent:

(i) \( f \) is differentiable at \( x_0 \).
(ii) There is a linear function \( T(x) \) and a function \( \varepsilon(x) \) on \((a, b)\) such that \( \lim_{x \to x_0} \varepsilon(x) = 0 \) and \( f(x) = T(x) + \varepsilon(x)(x - x_0) \).
(iii) There is a function \( \varphi(x) \) on \([a, b]\) such that \( f(x) = f(x_0) + \varphi(x)(x - x_0) \) and \( \lim_{x \to x_0} \varphi(x) = 0 \).

THM: “Arithmetic of Derivatives” Let \( f \) and \( g \) be differentiable functions at the point \( a \). Each of the functions \( cf \) (\( c \) a constant), \( f + g \), \( fg \), and \( f/g \) are differentiable at \( a \), except \( f/g \) if \( g(a) = 0 \). The formulas are:

(i) \( (cf)'(a) = c \cdot f'(a) \)
(ii) \( (f + g)'(a) = f'(a) + g'(a) \)
(iii) “Product rule” \( (fg)'(a) = f(a)g'(a) + f'(a)g(a) \)
(iv) “Quotient rule” \( (f/g)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2} \) if \( g(a) \neq 0 \).

THM: “The Chain Rule” Suppose that \( f \) is defined on \([a, b]\) and has range contained in \([c, d]\). Let \( g \) be defined on \([c, d]\). Suppose that \( f \) is differentiable at \( x_0 \in [a, b] \) and \( g \) is differentiable at \( f(x_0) \). Then the composition \( h(x) = g(f(x)) \) is defined, and \( h'(x_0) = g'(f(x_0))f'(x_0) \).
E.G.: The class of functions \( f(x) = x^n \sin(1/x) \) for \( x > 0 \) and \( f(0) = 0 \) for \( a > 0 \) is differentiable on \([0, \infty)\), but the derivative function is not continuous at \(0\).\(^{126}\)

**DEF:** A function \( f(x) \) is even if \( f(-x) = f(x) \) and odd when \( f(-x) = -f(x) \).\(^{127}\)

**THM:** “Fermat’s Theorem” Let \( f \) be a continuous function on an interval \([a, b]\) that takes its maximum or minimum value at a point \( x_0 \). Then, exactly one of the following holds:

(i) \( x_0 \) is an endpoint \( a \) or \( b \),
(ii) \( f \) is not differentiable at \( x_0 \),
(iii) \( f \) is differentiable at \( x_0 \) and \( f'(x_0) = 0 \).\(^{128}\)

**THM:** “Rolle’s Theorem” Suppose that \( f \) is a function that is continuous on \([a, b]\) and differentiable on \((a, b)\) such that \( f(a) = f(b) = 0 \). Then there is a point \( c \in (a, b) \) such that \( f'(c) = 0 \).\(^{129}\)

**THM:** “Mean Value Theorem” Suppose that \( f \) is a function that is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there is a point \( c \in (a, b) \) such that \( f'(C) = \frac{f(b) - f(a)}{b - a} \).\(^{130}\)

**COR:** Let \( f \) be a differentiable function on \([a, b]\).\(^{131}\)

(i) If \( f'(x) \) is (strictly) positive, then \( f \) is (strictly) increasing.
(ii) If \( f'(x) \) is (strictly) negative, then \( f \) is (strictly) decreasing.\(^{132}\)
(iii) If \( f'(x) = 0 \) at every \( x \in (a, b) \), then \( f(x) \) is constant.\(^{133}\)
(iv) If \( g \) is differentiable on \([a, b]\) with \( g'(x) = f'(x) \), then there is a constant \( c \) such that \( f(x) = g(x) + c \).\(^{134}\)

**DEF:** If a twice differentiable function \( f \) has \( f''(x) \) positive on an interval \([a, b]\) then \( f \) is convex or concave up. If \( f''(x) \) is negative, then \( f \) is concave or concave down. The points where \( f''(x) \) changes sign are called inflection points.\(^{135}\)

**THM:** “Darboux’s Theorem/Intermediate Value Theorem for Derivatives” If \( f \) is differentiable on \([a, b]\) and \( f'(a) < L < f'(b) \), then there is a point \( x_0 \) in \((a, b)\) at which \( f'(x_0) = L \).\(^{136}\)

**THM:** Let \( f \) be a one-to-one continuous function on an open interval \( I \), and let \( J = f(I) \). If \( f \) is differentiable at \( x_0 \in I \) and if \( f'(x_0) \neq 0 \), then \( f^{-1} \) is differentiable at \( y_0 = f(x_0) \) and \( (f^{-1})'(y_0) = \frac{1}{f'(x_0)} \).\(^{137}\)

### 6 Integration

**DEF:** Let \( f : [a, b] \to \mathbb{R} \) be a bounded function.\(^{138}\)

(i) A partition of \([a, b]\) is a finite set \( P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\} \). Set \( \Delta_j = x_j - x_{j-1} \) and define the mesh of a partition \( P \) as \( \text{mesh}(P) = \max_{1 \leq j \leq n} \Delta_j \).\(^{139}\)

(ii) Let the maximum and minimum be \( M_j(f, P) = \sup \{ f(x) \mid x_{j-1} \leq x \leq x_j \} \) and \( m_j(f, P) = \inf \{ f(x) \mid x_{j-1} \leq x \leq x_j \} \).

(iii) Let the upper (Darboux) sum and lower (Darboux) sum be \( U(f, P) = \sum_{j=1}^{n} M_j(f, P) \Delta_j \) and \( L(f, P) = \sum_{j=1}^{n} m_j(f, P) \Delta_j \).

(iv) If given an evaluation sequence \( X = \{x_j' \mid 1 \leq j \leq n\} \) with \( x_j' \in [x_{j-1}, x_j] \) the Riemann sum is \( I(f, P, X) = \sum_{j=1}^{n} f(x_j') \Delta_j \).\(^{140}\)

(v) A partition \( R \) is a refinement of a partition \( P \) provided \( P \subseteq R \). If \( P \) and \( Q \) are partitions, then \( R \) is a common refinement if \( P \cup Q \subseteq R \).

**LMA:** If \( R \) is a refinement of \( P \), then \( L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, P) \).\(^{141}\)

**COR:** If \( P \) and \( Q \) are any two partitions of \([a, b]\), \( L(f, P) \leq L(f, Q) \).\(^{142}\)

**DEF:** Define the lower Darboux integral, \( L(f) = \sup_{P} L(f, P) \) and the upper darboux integral, \( U(f) = \inf_{P} U(f, P) \). Note that \( L(f) \leq U(f) \).\(^{143}\) A bounded function \( f \) on a finite interval \([a, b]\) is called Riemann integrable if \( L(f) = U(f) \).

In this case, we write \( I(f) = \int_a^b f(x) \, dx = U(f) \).\(^{144}\)

**THM:** “Riemann’s Condition” Let \( f(x) : [a, b] \to \mathbb{R} \) be a bounded function. The following are equivalent:

(i) \( f \) is Riemann integrable.
(ii) For each \( \varepsilon > 0 \), there is a partition \( P \) so that \( U(f, P) - L(f, P) < \varepsilon \).\(^{145}\)

**COR:** Let \( f \) be a bounded real-valued function on \([a, b]\). If there is a sequence of partitions of \([a, b]\), \( P_n \) so that \( \lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0 \), then \( f \) is Riemann integrable. Moreover, if \( X_n \) is any choice of points \( x_{n,j} \) selected from each interval of \( P_n \), then \( \lim_{n \to \infty} I(f, P_n, X_n) = \int_a^b f(x) \, dx \).

**THM:** Let \( f(x) : [a, b] \to \mathbb{R} \) be a bounded function. The following are equivalent.\(^{146}\)

(i) \( f \) is Riemann integrable.
(ii) For each \( \varepsilon > 0 \), there is a partition \( P \) so that \( U(f, P) - L(f, P) < \varepsilon \).
(iii) “Cauchy Criterion for Integrability” For every \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that every partition \( Q \) such that \( \text{mesh}(Q) < \delta \) satisfies \( U(f, Q) - L(f, Q) < \varepsilon \).\(^{147}\)

**Mike Janssen**
7 Riemann-Stieltjes Integration

**Def:** Consider bounded functions \( f, g : [a, b] \to \mathbb{R} \). Given a partition of \([a, b], P = \{a = x_0 < x_1 < \cdots < x_n = b\}\), and an evaluation sequence \( X = (x'_1, x'_2, \cdots, x'_n) \) for \( P \), we define the **Riemann-Stieltjes sum**, or briefly, the **R-S sum**, for \( f \) w.r.t. \( g \) using \( P \) and \( X \), as

\[
I_g(f, P, X) = \sum_{j=1}^{n} f(x'_j)[g(x_j) - g(x_{j-1})].
\]

We say \( f \) is **Riemann-Stieltjes integrable** with respect to \( g \), denoted \( f \in \mathcal{R}(g) \), if there is a number \( L \) so that for all \( \varepsilon > 0 \), there is a partition \( P_{\varepsilon} \) so that, for all partitions \( P \) with \( P \geq P_{\varepsilon} \) and all evaluation sequences \( X \) for \( P \), we have

\[
|I_g(f, P, X) - L| < \varepsilon.
\]

**Thm:** If \( f \in \mathcal{R}(g) \) and \( f_1 \in \mathcal{R}(g) \) on \([a, b] \), and \( c_1, c_2 \in \mathbb{R} \), then \( c_1 f_1 + c_2 f_2 \in \mathcal{R}(g) \) on \([a, b] \) and

\[
\int_a^b c_1 f_1 + c_2 f_2 \, dg = c_1 \int_a^b f_1 \, dg + c_2 \int_a^b f_2 \, dg.
\]

Mike Janssen
Finally, if \( a < b < c \) and \( f \) is R-S integrable w.r.t. \( g \) on both \([a, b]\) and \([b, c]\), then it is R-S integrable w.r.t. \( g \) on \([a, c]\) and
\[
\int_a^c f \, dg = \int_a^b f \, dg + \int_b^c f \, dg.
\]

**Thm:** Suppose \( f, g : [a, b] \to \mathbb{R} \) are bounded and \( \varphi : [c, d] \to [a, b] \) is a strictly increasing, continuous function onto \([a, b]\). Let \( F = f \circ \varphi \) and \( G = g \circ \varphi \). If \( f \in \mathcal{R}(g) \) on \([a, b]\) then \( F \in \mathcal{R}(G) \) on \([c, d]\) and
\[
\int_c^d F \, dG = \int_a^b f \, dg.
\]

**Thm:** Let \( f, g : [a, b] \to \mathbb{R} \) be bounded functions. If \( f \in \mathcal{R}(g) \) on \([a, b]\), then \( g \in \mathcal{R}(f) \) on \([a, b]\) and
\[
\int_a^b f \, dg + \int_a^b g \, df = g(b)f(b) - g(a)f(a).
\]

**Thm:** If \( f : [a, b] \to \mathbb{R} \) has \( f \in \mathcal{R}(g) \), where \( g : [a, b] \to \mathbb{R} \) is \( C^1 \), then \( fg' \) is Riemann integrable on \([a, b]\) and
\[
\int_a^b f \, dg = \int_a^b f(x)g'(x) \, dx.
\]

**Def:** Let \( f \) and \( g \) be bounded functions on \([a, b]\). As usual, let \( P = \{x_0 < x_1 < \cdots < x_n\} \) be a partition of \([a, b]\) and recall that
\[
M_j(f, P) = \sup_{x_{j-1} \leq x \leq x_j} f(x) \quad \text{and} \quad m_j(f, P) = \inf_{x_{j-1} \leq x \leq x_j} f(x).
\]
Define the **upper sum with respect** to \( g \) and the **lower sum with respect** to \( g \) to be
\[
U_g(f, P) = \sum_{i=1}^n M_i(f, P)[g(x_i) - g(x_{i-1})]
\]
\[
L_g(f, P) = \sum_{i=1}^n m_i(f, P)[g(x_i) - g(x_{i-1})].
\]

**Rmrk:** If \( g \) is increasing then, for all evaluation sequences \( X \), \( L_g(f, P, X) \leq U_g(f, P) \).

**Lma:** (Refinement Lemma) Let \( f, g \) be bounded functions on \([a, b]\) and \( P, R \) partitions of \([a, b]\). Assume that \( g \) is increasing. If \( R \) is a refinement of \( P \), then
\[
L_g(f, P) \leq L_g(f, R) \leq U_g(f, R) \leq U_g(f, P).
\]

**Cor:** Let \( f, g \) be bounded functions on \([a, b]\) and assume that \( g \) is increasing. If \( P \) and \( Q \) are any two partitions of \([a, b]\), then
\[
L_g(f, P) \leq U_g(f, Q).
\]

**Def:** We define \( L_g(f) = \sup_P L_g(f, P) \) and \( U_g(f) = \inf_P U_g(f, P) \).

**Prop:** (Riemann-Stieltjes Condition) Let \( f, g \) be bounded functions on \([a, b]\) and assume that \( g \) is increasing on \([a, b]\). TFAE:
1. \( f \) is R-S integrable w.r.t. \( g \).
2. \( U_g(f) = L_g(f) \), and
3. for each \( \varepsilon > 0 \), there is a partition \( P \) so that \( U_g(f, P) - L_g(f, P) < \varepsilon \).

**Def:** Given a function \( f : [a, b] \to \mathbb{R} \) and a partition \([a, b]\), say \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \), then the **variation** of \( f \) over \( P \) is
\[
V(f, P) := \sum_{i=1}^n |f(x_i) - f(x_{i-1})|.
\]

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Mike Janssen
8 NORMED VECTOR SPACES

Analysis Study Guide

RMKR: If \( P \) is a partition of \([a, b]\) and \( Q \) is a partition of \([b, c]\), then for a function \( f : [a, c] \to \mathbb{R} \), \( V(f, P \cup Q) = V(f, P) + V(f, Q) \).

RMKR: If \( R \) is a refinement of \( P \), then \( V(f, P) \leq V(f, R) \).

DEF: The (total) variation of \( f \) on \([a, b]\) is

\[
V^b_a f = \sup \{ V(f, P) : P \text{ a partition of } [a, b] \}.
\]

We say \( f \) is of bounded variation on \([a, b]\) if \( V^b_a f \) is finite.

LMA: If \( a < b < c \) and \( f : [a, c] \to \mathbb{R} \) is given, then

\[
V^c_a f = V^b_a f + V^c_b f.
\]

THM: If \( f : [a, b] \to \mathbb{R} \) is of bounded variation on \([a, b]\), then there are increasing functions \( g, h : [a, b] \to \mathbb{R} \) so that \( f = g - h \).

THM: If \( f : [a, b] \to \mathbb{R} \) is bounded by \( M \), \( g : [a, b] \to \mathbb{R} \) is of bounded variation and \( f \in \mathcal{R}(g) \) on \([a, b]\), then

\[
\left| \int_a^b f dg \right| \leq M \cdot V^b_a g.
\]

THM: If \( f : [a, b] \to \mathbb{R} \) is continuous and \( g : [a, b] \to \mathbb{R} \) is increasing, then \( f \in \mathcal{R}(g) \) on \([a, b]\).

COR: If \( f : [a, b] \to \mathbb{R} \) is continuous and \( g : [a, b] \to \mathbb{R} \) is of bounded variation, then \( f \in \mathcal{R}(g) \) on \([a, b]\).

8 Normed Vector Spaces

DEF: Let \( V \) be a vector space over \( \mathbb{R} \). A norm on \( V \) is a function \(|\cdot|\) on \( V \) taking values in \([0, +\infty)\) with the following properties:

1. (positive definite) \(|x| = 0\) if and only if \( x = 0 \),
2. (homogeneous) \(|\alpha x| = |\alpha| |x|\) for all \( x \in V \) and \( \alpha \in \mathbb{R} \), and
3. (triangle inequality) \(|x + y| \leq |x| + |y|\) for all \( x, y \in V \).

E.G.: Let \( V = \mathbb{R}^n \). Then the following are norms:

\[
|\mathbf{x}| = ||(x_1, \ldots, x_n)||_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}
\]

\[
|\mathbf{x}|_1 = ||(x_1, \ldots, x_n)||_1 = \sum_{i=1}^n |x_i|
\]

\[
|\mathbf{x}|_\infty = ||(x_1, \ldots, x_n)||_\infty = \max_{1 \leq i \leq n} |x_i|.
\]

E.G.: Fix a real number \( p \) in \([1, \infty)\). Define the \( L^p[a, b] \) norm on \( C[a, b] \) by

\[
||f||_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}.
\]

DEF: A complete normed vector space \( V \) is called a Banach space.

PROP: A sequence \( x_n \) in a normed vector space \( V \) converges to a vector \( x \) if and only if for each open set \( U \) containing \( x \), there is an integer \( N \) so that \( x_n \in U \) for all \( n \geq N \).

DEF: A subset \( K \) of a normed vector space \( V \) is compact if every sequence \( (x_n) \) of points in \( K \) has a subsequence \( (x_{n_k}) \) which converges to a point in \( K \).

Mike Janssen
**8.1 Inner Product Spaces**

**Def:** An inner product on a vector space \( V \) is a function \( \langle x, y \rangle \) on pairs \( (x, y) \) of vectors in \( V \times V \) taking values in \( \mathbb{R} \) satisfying the following properties:

1. (positive definiteness) \( \langle x, x \rangle \geq 0 \) for all \( x \in V \) and \( \langle x, x \rangle = 0 \) only if \( x = 0 \).
2. (symmetry) \( \langle x, y \rangle = \langle y, x \rangle \) for all \( x, y \in V \).
3. (bilinearity) For all \( x, y, z \in V \) and scalars \( \alpha, \beta \in \mathbb{R} \), \( \langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \).

Given an inner product space, it is easy to check that the following definition gives us a norm:

\[
||x|| = \langle x, x \rangle^{1/2}.
\]

**E.g.:** The space \( C[a, b] \) can be given an inner product

\[
\langle f, g \rangle = \int_a^b f(x)g(x)dx,
\]

which gives rise to the \( L^2 \) norm.

**Thm:** (Cauchy-Schwarz Inequality) For all vectors \( x, y \) in an inner product space \( V \),

\[
|\langle x, y \rangle| \leq ||x|| \cdot ||y||.
\]

Equality holds if and only if \( x \) and \( y \) are colinear.

**Cor:** Let \( V \) be an inner product space with induced norm \( ||\cdot|| \). Then the inner product is continuous (i.e., if \( x_n \) converges to \( x \) and \( y_n \) converges to \( y \), then \( \langle x_n, y_n \rangle \) converges to \( \langle x, y \rangle \)).

**8.2 Orthonormal Sets**

**Def:** Two vectors \( x \) and \( y \) are called orthogonal if \( \langle x, y \rangle = 0 \). A collection of vectors \( \{e_n : n \in S\} \) in \( V \) is called orthonormal if \( ||e_n|| = 1 \) for all \( n \in S \) and \( \langle e_n, e_m \rangle = 0 \) for \( n \neq m \in S \). This set is called an orthonormal basis if in addition this set is maximal with respect to being an orthonormal set.

**Prop:** An orthonormal set is linearly independent. An orthonormal basis for a finite-dimensional inner product space is a basis.

**Lma:** The functions \( \{1, \sqrt{2}\cos n\theta, \sqrt{2}\sin n\theta : n \geq 1\} \) form an orthonormal set in \( C[-\pi, \pi] \) with the inner product

\[
\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)g(\theta)d\theta.
\]

**Def:** A trigonometric polynomial is a finite sum \( f(\theta) = A_0 + \sum_{k=1}^{N} A_k \cos k\theta + B_k \sin k\theta \).

**Def:** Denote the Fourier series of \( f \in C[-\pi, \pi] \) by \( f = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta \), where \( A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \), and for \( n \geq 1 \), \( A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt \) and \( B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt \).

**Lma:** Let \( \{e_1, \cdots, e_n\} \) be an orthonormal set in an inner product space \( V \). If \( M \) is the subspace spanned by \( \{e_1, \cdots, e_n\} \), then every vector \( x \in M \) can be written uniquely as \( \sum_{i=1}^{n} \alpha_i e_i \), where \( \alpha_i = \langle x, e_i \rangle \). In other words, the set \( \{e_1, \cdots, e_n\} \) is linearly independent. Moreover, for each \( y \in v \) with \( \langle y, e_i \rangle = \beta_i \), and each \( x = \sum_{j=1}^{n} \alpha_j e_j \) in \( M \), \( \langle x, y \rangle = \sum_{i=1}^{n} \alpha_i \beta_i \).

**Cor:** If \( V \) is an inner product space of finite dimension \( n \), then it has an orthonormal basis \( \{e_i : 1 \leq i \leq n\} \) and the inner product is given by

\[
\left\langle \sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j \right\rangle = \sum_{i=1}^{n} \alpha_i \beta_i.
\]
8.3 Orthogonal Expansions in IPS

**DEF:** A projection is a linear map $P$ such that $P^2 = P$. In addition, say that $P$ is an orthogonal projection if

$\text{ker } P = \text{Ran } (I - P)$ because $Px = 0$ if and only if $(I - P)x = x$. Further, $x = (I - P)y$ implies $Px = (P - P^2)y = 0$. Thus, when $P$ is an orthogonal projection, the vectors $Px$ and $(I - P)x$ are orthogonal.

**Theorem (Projection Theorem)** Let $A = \{e_1, \ldots, e_n\}$ be an orthonormal set in an IPS $V$ and let $M$ be the subspace spanned by $A$. Define $P : V \to M$ by

$$P = \sum_{j=1}^n \langle y, e_j \rangle e_j,$$

for each $y \in V$. Then $P$ is the orthogonal projection onto $M$ and

$$\|y\| = \sum_{j=1}^n \langle y, e_j \rangle^2.$$

Moreover, for all $v \in M$,

$$\|y - v\|^2 = \|y - Py\|^2 + \|Py - v\|^2.$$

In particular, $Py$ is the closest vector in $M$ to $y$.

**Proposition (Bessel’s Inequality)** Let $\{e_n : n \in S\}$ be an orthonormal set in an inner product space $V$. For each vector $x \in V$, $\sum_{n \in S} |\langle x, e_n \rangle|^2 \leq \|x\|^2$.

**Definition:** A complete inner product space is called a Hilbert space.

**Example:** The space $\ell^2$ consists of all sequences $x = (x_n)_{n=1}^\infty$ such that $\|x\|_2 = \left( \sum_{n=1}^\infty x_n^2 \right)^{1/2}$ is finite. The inner product on $\ell^2$ is given by $\langle x, y \rangle = \sum_{n=1}^\infty x_ny_n$.

**Theorem:** The space $\ell^2$ is complete.

**Proof.** Forthcoming.

**Theorem (Parseval’s Theorem)** Let $S \subseteq \mathbb{N}$ and $E = \{e_n : n \in S\}$ be an orthonormal set in a Hilbert space $H$. Then the subspace $M = \text{span } \{E\}$ consists of all vectors $x = \sum_{n \in S} \alpha_n e_n$, where the coefficient sequence $(\alpha_n)_{n=1}^\infty$ belongs to $\ell^2$.

Further, if $x$ is a vector in $H$, then $x \in M$ if and only if

$$\sum_{n \in S} |\langle x, e_n \rangle|^2 = \|x\|^2.$$

**Corollary:** Let $E = \{e_n : n \in S\}$ be an orthonormal set in a Hilbert space $H$. Then there is a continuous linear orthogonal projection $P_E$ of $H$ onto $M = \text{span } \{E\}$ given by $P_E(x) = \sum_{n \in S} \langle x, e_n \rangle e_n$.

**Corollary:** If $E = \{e_i : i \geq 1\}$ is an orthonormal basis for a Hilbert space $H$, every vector $x \in H$ may be uniquely expressed as $x = \sum_{i=1}^\infty \alpha_i e_i$, where $\alpha_i = \langle x, e_i \rangle$.

8.4 Finite-Dimensional Normed Spaces

**Lemma:** If $\{v_1, \ldots, v_n\}$ is a LI set in a normed vector space $(V, \|\|)$, then there exist positive constants $0 < c < C$ such that for all $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we have $c \|a\|_2 \leq \left| \sum_{i=1}^n a_i v_i \right| \leq C \|a\|_2$.

**Remark:** This effectively means that every finite-dimensional normed vector space has the same topology as $(\mathbb{R}^n, \|\|_2)$ in the sense that they have the “same” convergent sequences, the same open/closed sets, etc.

**Corollary:** A subset of a finite-dimensional normed vector space is compact if and only if it is closed and bounded.

**Corollary:** A finite-dimensional subspace of a normed vector space is complete, and in particular it is closed.

**Theorem:** Let $(V, \|\|)$ be a normed vector space, and let $W$ be a finite-dimensional subspace of $V$. Then for any $v \in V$, there is at least one closest point $w^* \in W$ so that $\|v - w^*\| = \inf \{\|v - w\| : w \in W\}$. 

Mike Janssen
9 Limits of Functions

9.1 Limits of Functions

**Def:** Let \((f_n)\) be a sequence of functions from \(S \subset \mathbb{R}^n\) into \(\mathbb{R}^m\). This sequence **converges pointwise** to a function \(f\) if
\[
\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for all } x \in S.
\]

**Def:** Let \((f_n)\) be a sequence of functions from \(S \subset \mathbb{R}^n\) to \(\mathbb{R}^m\). This sequence **converges uniformly** to a function \(f\) if for every \(\varepsilon > 0\), there is an integer \(N\) so that
\[ ||f_n(x) - f(x)|| < \varepsilon \quad \text{for all } s \in S \text{ and } n \geq N. \]

**Thm:** For a sequence of functions \((f_n)\) in \(C_b(S, \mathbb{R}^m)\) (the space of all bounded continuous functions from \(S\) to \(\mathbb{R}^m\)), \((f_n)\) converges uniformly to \(f\) if and only if
\[ \lim_{n \to \infty} ||f_n - f||_\infty = 0. \]

E.g.: The sequence \(f_n(x) = x^n\) for \(x \in [0, 1]\) converges pointwise to \(\chi_{\{1\}}\). The functions \(f_n\) are polynomials, and hence not only continuous but even smooth; while the limit function has a discontinuity at the point 1. For each \(n \geq 1\), we have \(f_n(1) = 1\) and so
\[ ||f_n - \chi_{\{1\}}||_\infty = \sup_{0 \leq x < 1} |x^n - 0| = 1. \]

So \(f_n\) does not converge in the uniform norm. Indeed, to contradict the definition, take \(\varepsilon = 1/2\). For each \(n\), let \(x_n = 2^{-1/2n}\). Then
\[ |f_n(x_n) - \chi_{\{1\}}(x_n)| = \frac{1}{\sqrt{2}} > \varepsilon. \]

Hence no integer \(N\) satisfies the definition.

9.2 Uniform Convergence and Continuity

**Thm:** Let \((f_n)\) be a sequence of continuous functions mapping a subset of \(S\) of \(\mathbb{R}^n\) into \(\mathbb{R}^m\) that converges uniformly to a function \(f\). Then \(f\) is continuous.

**Thm:** (Completeness of \(C(K)\)) If \(K\) is a compact set, the space \(C(K)\) of all continuous functions on \(K\) with the sup norm is complete.

9.3 Uniform Convergence and Integration

**Thm:** (Integral Convergence Theorem) Let \((f_n)\) be a sequence of continuous functions on the closed interval \([a, b]\) converging uniformly to \(f(x)\) and fix \(c \in [a, b]\). Then the functions
\[ F_n(x) = \int_c^x f_n(t)\,dt \quad \text{for } n \geq 1 \]
converge uniformly on \([a, b]\) to the function \(F(x) = \int_c^x f(t)\,dt\).

**Cor:** Suppose that \((f_n)\) is a sequence of continuously differentiable functions on \([a, b]\) such that \((f'_n)\) converges uniformly to a function \(g\) and there is a point \(c \in [a, b]\) so that \(\lim_{n \to \infty} f_n(c) = \gamma\) exists. Then \((f_n)\) converges uniformly to a differentiable function \(f\) with \(f(c) = \gamma\) and \(f' = g\).

**Prop:** Let \(f(x, t)\) be a continuous function on \([a, b] \times [c, d]\). Define \(F(x) = \int_c^d f(x, t)\,dt\). Then \(F\) is continuous on \([a, b]\).

**Thm:** (Leibniz’s Rule) Suppose that \(f(x, t)\) and \(\frac{\partial}{\partial x} f(x, t)\) are continuous functions on \([a, b] \times [c, d]\). Then \(F(x) = \int_c^d f(x, t)\,dt\) is differentiable and
\[ F'(x) = \int_c^d \frac{\partial}{\partial x} f(x, t)\,dt. \]
9.4 Series of Functions

**Thm:** Let \((f_n)\) be a sequence of continuous functions from a subset \(S\) of \(\mathbb{R}^n\) into \(\mathbb{R}^m\). If \(\sum_{n=1}^{\infty} f_n(x)\) converges uniformly, then it is continuous.

**Def:** Let \(S \subset \mathbb{R}^n\). We say that a series of functions \(f_k\) from \(S\) to \(\mathbb{R}^m\) is uniformly Cauchy on \(S\) if for every \(\varepsilon > 0\) there is \(N\) so that

\[
\left| \sum_{i=k+1}^{l} f_i(x) \right| \leq \varepsilon \quad \text{whenever} \quad x \in S \quad \text{and} \quad l > k \geq N.
\]

**Thm:** A series of functions converges uniformly if and only if it is uniformly Cauchy.

**Thm:** (Weierstrass M-Test) Suppose that \(a_n(x)\) is a sequence of functions on \(S \subset \mathbb{R}^k\) into \(\mathbb{R}^m\) and \((M_n)\) is a sequence of real numbers so that

\[
||a_n||_\infty = \sup_{x \in S} ||a_n(x)|| \leq M_n \quad \text{for all} \quad x \in S.
\]

If \(\sum_{n \geq 1} M_n\) converges, then the series \(\sum_{n \geq 1} a_n(x)\) converges uniformly on \(S\).

**E.g.:** The function \(f(x) = \sum_{k \geq 1} 2^{-k} \cos(10^k \pi x)\) is continuous on \(\mathbb{R}\) and differentiable at no point in \(\mathbb{R}\).

9.5 Power Series

**Def:** A power series is a series of functions of the form

\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.
\]

**Thm:** (Hadamard’s Theorem) Given a power series \(\sum_{n=0}^{\infty} a_n x^n\) there is \(R \in [0, +\infty) \cup \{+\infty\}\) so that the series converges for all \(x\) with \(|x| < R\) and diverges for all \(x\) with \(|x| > R\). Moreover, the series converges uniformly on each closed interval \([a, b]\) contained in \((-R, R)\). Finally, if \(\alpha = \limsup_{n \to \infty} |a_n|^{1/n}\), then

\[
R = \begin{cases} 
+\infty & \text{if} \ \alpha = 0 \\
0 & \text{if} \ \alpha = +\infty \\
\frac{1}{\alpha} & \text{if} \ \alpha \in (0, +\infty).
\end{cases}
\]

We call \(R\) the radius of convergence of the power series.

**Thm:** (Term-by-Term Differentiation of Series) If \(f(x) = \sum_{n=0}^{\infty} a_n x^n\) has radius of convergence \(R > 0\), then \(\sum_{n=1}^{\infty} na_n x^{n-1}\) has radius of convergence \(R\), \(f\) is differentiable on \((-R, R)\) and, for \(x \in (-R, R)\),

\[
f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}.
\]

Further, \(\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}\) has radius of convergence \(R\) and, for \(x \in (-R, R)\),

\[
\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.
\]

9.6 Compactness and Subsets of \(C(K)\)

**Def:** A family of functions \(\mathcal{F}\) mapping a set \(S \subset \mathbb{R}^n\) into \(\mathbb{R}^m\) is equicontinuous at a point \(a \in S\) if for every \(\varepsilon > 0\), there is an \(r > 0\) such that

\[
||f(x) - f(a)|| < \varepsilon \quad \text{whenever} \quad ||x - a|| < r \quad \text{and} \quad f \in \mathcal{F}.
\]

Mike Janssen
The family $\mathcal{F}$ is **equicontinuous** on a set $S$ if it is equicontinuous at every point in $S$. The family $\mathcal{F}$ is **uniformly equicontinuous** on $S$ if for each $\varepsilon > 0$, there is an $r > 0$ such that

$$||f(x) - f(y)|| < \varepsilon \quad \text{whenever} \quad ||x - y|| < r, \ x, y \in S \text{ and } f \in \mathcal{F}.$$ 

**LMA:** Let $K$ be a compact subset of $\mathbb{R}^n$. A compact subset $\mathcal{F}$ of $C(K, \mathbb{R}^m)$ is equicontinuous.

**PROP:** If $\mathcal{F}$ is an equicontinuous family of functions on a compact set, then it is uniformly equicontinuous.

**DEF:** A subset $S$ of $K$ is called an $\varepsilon$-net of $K$ if

$$K \subset \bigcup_{a \in S} B_\varepsilon(a).$$

A set $K$ is **totally bounded** if it has a finite $\varepsilon$-net for every $\varepsilon > 0$.

**LMA:** Let $K$ be a bounded subset of $\mathbb{R}^m$. Then $K$ is totally bounded, man.

**COR:** Let $K$ be a compact subset of $\mathbb{R}^m$. Then $K$ contains a sequence $\{x_i : i \geq 1\}$ that is dense in $K$. Moreover, for any $\varepsilon > 0$, there is an integer $N$ so that $\{x_i : 1 \leq i \leq N\}$ forms an $\varepsilon$-net for $K$.

**THM:** (Arzela-Ascoli) Let $K$ be a compact subset of $\mathbb{R}^n$. A subset $\mathcal{F}$ of $C(K, \mathbb{R}^m)$ is compact if and only if it is closed, bounded, and equicontinuous.

## 10 Metric Spaces

### 10.1 Definitions and Examples

**DEF:** Let $X$ be a set. A **metric** on a set $X$ on a set $X$ is a function $d : X \times X \to [0, \infty)$ with the following properties:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A **metric space** is a set $X$ with a metric $d$, denoted $(X, d)$.

**E.G.:** If $X$ is a subset of a normed space $V$, define $d(x, y) = ||x - y||$.

**E.G.:** Define a metric on $\mathbb{Z}$ by $p_2(n, m) = 0$ and $p_2(m, n) = 2^{-d}$, where $d$ is the largest power of 2 dividing $m - n$.

**E.G.:** If $X$ is a closed subset of $\mathbb{R}^n$, let $K(X)$ denote the collection of all nonempty compact subsets of $X$. If $A$ is a compact subset of $X$ and $x \in X$, we define $\text{dist}(x, A) = \inf_{a \in A} ||x - a||$. Then we define the **Hausdorff metric** on $K(X)$ by

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\} = \max \left\{ \sup_{a \in A} \inf_{b \in B} ||a - b||, \sup_{b \in B} \inf_{a \in A} ||a - b|| \right\}.$$ 

**PROP:** When $X \subseteq \mathbb{R}^n$ is closed, then $(K(X), d_H)$ is a complete metric space.

**THM:** Let $f : (X, \rho) \to (Y, \sigma)$. TFAE:

1. $f$ is continuous on $X$;
2. for every convergent sequence $(x_n)_{n=1}^\infty$ with $\lim x_n = a$ in $X$, we have $\lim f(x_n) = f(a)$; and
3. $f^{-1}(U)$ is open for every open set $U \subseteq Y$.

**THM:** The space $C_b(X)$ of all bounded continuous functions on $X$ with the sup norm $||f|| = \sup \{|f(x)| : x \in X\}$ is complete.

**RMRK:** Given a metric space $(X, d)$, $(X, \overline{d})$ (the bounded metric space) has the same topology as $(X, d)$.

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*Mike Janssen*


10.2 Compact Metric Spaces

Thm: (Borel-Lebesgue) For a metric space \( X \), TFAE:

1. \( X \) is compact.
2. Every collection of closed subsets of \( X \) with the FIP has nonempty intersection.
3. \( X \) is sequentially compact.
4. \( X \) is complete and totally bounded.

Prop: A metric space \( X \) is complete if and only if every decreasing sequence of nonempty closed sets (balls) with radii going to zero has nonempty intersection.

11 Discrete Dynamical Systems

11.1 Fixed Points and the Contraction Principle

Def: For a metric space \((X, d)\), call \( T : X \to X \) a contraction if there is \( c \in (0, 1) \) such that \( d(Tx, Ty) \leq cd(x, y) \) for all \( x, y \in X \). That is, \( T \) satisfies a Lipschitz condition with \( c < 1 \). For \( T : X \to X \) call \( x_0 \in X \) a fixed point for \( T \) if \( Tx_0 = x_0 \).

E.g.: Let \( X = \mathbb{R}, T(x) = mx + b \) (an affine map). If \( m = 0, b \) is a fixed point and \( \text{im} T = B \). If \( b = 0 \) then \( T(x) = mx \) and \( x = 0 \) is a fixed point. If \( m \neq 1 \), this is the only fixed point. If \( |m| < 1 \), then for any \( x \in T, T^n(x) \to x \). If \( |m| > 1 \), then \( T^n(x) \) diverges. If \( m = -1 \), then \( T(T(x)) = T(-x) = x \) so for every \( x \neq 0 \), we have a period 2 orbit.

For general \( b \), we can see inductively that \( x_n = nm^nx + b(1 + m + \cdots + m^{n-1}) = nm^nx + b \left( \frac{1 - m^n}{1 - m} \right) \) for \( m \neq 1 \). If \( |m| < 1 \), then this converges to \( \frac{b}{1-m} \). If \( m > 1 \) and \( x \neq 0 \), this sequence diverges to \( \infty \). To find fixed points, we have \( x = T(x) = mx + b \) implies \( x = b/(1-m) \) for \( m \neq 1 \). If \( |m| > 1 \) and \( x_n = T(x_{n-1}) \), then \( x_{n-1} = T^{-1}(x_n) \). Now \( T^{-1} \) is an affine map with slope \( |1/m| < 1 \). Define \( x_{-n} = (T^{-1})^n(x) \), which converges to \( b/(1-m) \) as \( n \to \infty \). It follows that \( T^n(x) \) diverges for all \( x \neq b/(1-m) \). It turns out that \( T \) is a contraction whenever \( |m| < 1 \). Notice we have fixed points even for \( |m| \geq 1 \).

Thm: (Banach Contraction Principle) Let \((X, d)\) be a complete metric space. Let \( T : X \to X \) be a contraction. Then \( T \) has a unique fixed point \( x^* \), and for all \( x \in X, T^n(x) \to x^* \). Moreover for any \( x \in X, \)

\[
d(x^*, T^n(x)) \leq c^n d(x^*, x) \leq \frac{c^n}{1-c} d(x, T(x)).
\]

11.2 Iterated Function Systems

Def: We call a linear map \( T : \mathbb{R}^n \to \mathbb{R}^n \) a similitude if there is \( r > 0 \) such that for all \( x, y \in \mathbb{R}^n, ||Tx - Ty|| = r||x - y|| \). If \( r < 1 \), then \( T \) is a contraction, called a contractive similitude. An iterated function system (IFS) is a finite set of contractive similitudes.

Fact: If \( A_1, \ldots, A_r, B_1, \ldots, B_r \in K(X) \), then \( d_H(A_1 \cup \cdots \cup A_r, B_1 \cup \cdots \cup B_r) \leq \max \{ d_H(A_1, B_1), \ldots, d_H(A_r, B_r) \} \).

Thm: Let \( X \subseteq \mathbb{R}^n \) be closed. Suppose \( \{T_1, \ldots, T_r\} \) is an IFS on \( X \) where \( T_i \) has Lipschitz constant \( s_i \). Define \( T \) on \( K(X) \) by \( T(A) = T_1(A) \cup \cdots \cup T_r(A) \), and let \( s = \max \{ s_1, \ldots, s_r \} \).

1. \( T \) is a contraction on \( K(X) \) with Lipschitz constant \( s \),
2. There is a unique \( A \in K(X) \) such that \( A = T(A) \), and
3. For any \( B \in K(X), d_H(T^k(B), A) \leq s^k d_H(B, A) \leq s^k/(1-s) d_H(T(B), B) \).

E.g.: Concretely, how do we find the magical \( A \) for a given IFS \( \{T_1, \ldots, T_r\} \)? Consider words \( w \in \{1, \ldots, r\} \) and let \( |w| \) denote the length of \( w \). Given a word \( i_1i_2\cdots i_m = w \), define a map \( T_w : X \to X \) by \( T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m} \). Notice that \( T_w \) is a contractive similitude with constant \( s_{i_1} \cdots s_{i_m} \leq s^{|w|} \) (where the \( s_{i_j} \) are the constants for \( T_{i_j} \) respectively and \( s \) is their max). There are three approaches:

(1) Find \( B \in K(X) \) such that \( T(B) \subseteq B \). Then \( A = \cap_{k=1}^{\infty} T^k(B) \).

Mike Janssen
(2) For each word $w$, let $a_w$ be the fixed point of $T_w$. Then $A = \{a_w : w \text{ a word in } 1, \cdots, r\}$.

(3) Pick any point $a \in A$. Then $A = \{T_w(a) : w \text{ a word in } 1, \cdots, r\}$.

**Thm:** (Fixed points of IFS’s are dense in $K(\mathbb{R}^n)$) For any $C \subseteq \mathbb{R}^n$ compact and $\varepsilon > 0$, there is an IFS $\{T_1, \cdots, T_r\}$ so that its fixed point set $A$ satisfies $d_H(A, C) < \varepsilon$.

## 12 Approximation by Polynomials

### 12.1 Taylor Series

**Def:** For $f : [a, b] \rightarrow \mathbb{R}$, $n$ times differentiable at $c \in (a, b)$, the $n^{\text{th}}$ Taylor polynomial for $f$ at $c$ is

$$P_n(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + \cdots + f^{(n)}(c)\frac{(x-c)^n}{n!}.$$  

**Lma:** $P_n$ is the unique polynomial of degree at most $n$ such that $P_n^{(k)}(c) = f^{(k)}(c)$.

**Thm:** Suppose $f$ is $n+1$ times differentiable on $[a, b]$. Let $x, c \in (a, b)$. There is $d$ between $x$ and $c$ such that

$$f(x) - P_n(x) = f^{(n+1)}(d)\frac{(x-c)^{n+1}}{(n+1)!}.$$  

Furthermore, if $|f^{(n+1)}|$ is bounded by $M$ on $[a, b]$, then

$$|f(x) - P_n(x)| \leq M\frac{|x-c|^{n+1}}{(n+1)!}.$$  

**Def:** Formally, we say $f$ is $O(g)$ near $a \in \mathbb{R}$ if there is some $M > 0$, $\delta > 0$ such that $|f(x)| \leq M|g(x)|$ for all $x$ with $0 < |x-a| \leq \delta$.

**Prop:** Rules for computation: $O(f) \pm O(g) = O(\max\{f, g\})$, $O(f)O(g) = O(fg)$.

**E.g.:** $O((x-a)^m) \pm O((x-a)^n) = O((x-a)^{\min\{m, n\}})$, and $O((x-a)^m)((x-a)^n) = O((x-a)^{m+n})$.

**Rmrk:** Taylor’s Theorem says $O(f - P_n) = O((x-c)^{n+1})$.

### 12.2 How not to Approximate a Function

**Thm:** (Weierstrass Approximation Theorem) Let $f$ be any continuous real-valued function on $[a, b]$. Then there is a sequence of polynomials $p_n$ that converges uniformly to $f$ on $[a, b]$.

### 12.3 Bernstein’s Proof of the Weierstrass Theorem

**Def:** We define the Bernstein polynomials to be $B^n_k(x) = \binom{n}{k}x^k(1-x)^{n-k}$. We define a polynomial $B_n f$ by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B^n_k(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k}x^k(1-x)^{n-k}.$$  

**Prop:** The map $B_n$ is linear and monotone. That is, for all $f, g \in C[0, 1]$ and $\alpha \in \mathbb{R},$

1. $B_n(f + g) = B_n f + B_n g.$
2. $B_n(\alpha f) = \alpha B_n f$
3. $B_n f \geq 0$ if $f \geq 0$
4. $B_n f \geq B_n g$ if $f \geq g$
5. $|B_n f| \leq B_n g$ if $|f| \leq g.$
Notes

Donsig, 825 Notes, 08/29.
Donsig, 825 Notes, 08/29.
Donsig, 825 Notes, 08/31.
D&D Definition 2.3.1. p. 38.
D&D Theorem 2.3.6. p. 39.
D&D Proposition 2.4.2. p. 42; Ross, Theorem 9.1. p. 43.
D&D Theorem 2.4.3. p. 42; Ross, Thm 9.2-6. pp.43-46.
D&D Theorem 2.5.3. p. 46.
D&D Theorem 2.5.4. p. 47; Ross, Thm 10.2. p. 55.
Ross, Theorem 10.7. p. 58.
Ross, Definition 11.1. p. 63.
Ross, Theorem 11.2. p. 67.
Ross, Theorem 11.3. p. 67.
Ross, Corollary 11.4. p. 68.
Ross, Definition 11.6. p. 70.
Ross, Theorem 11.7. p. 71.
Ross, Theorem 11.8. p. 72.
D&D Lemma 2.6.3. p. 51.
D&D Theorem 2.6.4. p. 52.
Donsig, 825 Notes, 09/17.
D&D Definition 2.7.2. p. 56; Ross, Definition 10.8. p. 60.
D&D Definition 2.7.3. p. 56.
D&D Theorem 2.7.4. p. 56; Ross, Theorem 10.11. p. 61.
D&D Def 3.1.1. p. 66.
D&D Example 3.1.3. p. 68.
Donsig, 825 Notes, 09/19.
D&D Theorem 3.1.5. p. 69; Ross, Definition 14.3, Theorem 14.4. pp. 92-93.
D&D Prop 3.2.1.. p. 71.
D&D Theorem 3.2.2.. p. 71.
D&D Theorem 3.2.3. p. 71; Ross, Theorem 14.6. p. 93.
Ross, Theorem 14.8. p. 94.
D&D Theorem 3.2.4, p.72; Ross, Theorem 14.9. p. 94.
Donsig, 825 Notes, 09/26.
Donsig, 825 Notes, 09/26.
D&D Definition 3.2.5. p. 72.
D&D Theorem 3.2.6. p. 72.
D&D Definition 3.4.2. p. 81.
D&D Definition 3.4.4. p. 81.
D&D Lemma 3.4.7. p. 82.
D&D Theorem 3.4.8. p. 83.
D&D Lemma 3.4.9. p. 85.
D&D Theorem 3.4.10. p. 85.
D&D Problem 3.4.G. Solution in 825 Problem Set 5.
D&D p.89.
D&D Theorem 4.1.1. p.89.
D&D Theorem 4.1.2. p.89.
D&D p.91.
D&D Lemma 4.1.3. p.91.
D&D Definition 4.2.1. p.92.
D&D Lemma 4.2.2. p.93.
D&D Lemma 4.2.3. p.93.
D&D Definition 4.2.4. p.93.
D&D Theorem 4.2.5. p.94.
D&D Definition 4.3.1. p. 96.
D&D Proposition 4.3.3. p. 97.
D.S. 10/15.
D&D Definition 4.3.4. p.97.
D&D Definition 4.3.6. p.98.
D&D Proposition 4.3.5. p.97.
D&D Theorem 4.3.8. p.98.
D&D Proposition 4.3.9. p. 99.

Mike Janssen
13 Counterexamples

135 D&D p. 151
136 D&D 6.2.1 p. 152; Ross Thm 29.8 p. 217
137 Ross Thm 29.9 pp. 218-9
138 D&D Def 6.3.1 p. 154; Ross Def 32.1 pp. 243-4
139 Ross Def 32.6 p. 249
140 Ross Def 32.8 p. 250
141 D&D Lma 6.3.2 p. 155; Ross Lma 32.2 p. 246
142 D&D Cor 6.3.3 p. 155; Ross Lma 32.3 p. 247
143 Ross Thm 32.4 p. 247
144 D&D Def 6.3.4 p. 156; Ross Def 32.1 p. 244
145 D&D Thm 6.3.5 p. 156; Ross Thm 32.5 p. 248
146 D&D Thm 6.3.8 p. 158
147 Ross Thm 32.7 p. 249
148 D&D Thm 6.3.8, (4) p. 158

149 D&D Thm 6.3.9 p. 160; Ross Thm 33.1 p. 253
150 D&D Thm 6.3.10 p. 160; Ross Thm 33.2 p. 254
151 D&D Def 6.3.S p. 164
152 Ross Thm 33.3 p. 254
153 Ross Thm 33.4 p. 257
154 Ross Thm 33.5 p. 257
155 Ross Thm 33.6 p. 258
156 D&D Thm 6.4.1 p. 165; Ross Thm 34.1 p. 262
157 D&D p. 165
158 D&D Cor 6.4.2 p. 165; Ross Thm 34.3 p. 264
159 D&D Lma 6.4.3 p. 165
160 D&D p. 167; Ross Thm 34.2 p. 263
161 D&D p. 167; Ross Thm 34.4 p. 265
162 D&D 6.4.C p. 168; Ross Thm 33.9 p. 259

Mike Janssen
Notes